

Recursion Operators admitted by non-Abelian Burgers equations: Some Remarks

Sandra CARILLO (1 and 2), Mauro LO SCHIAVO (1) and Cornelia SCHIEBOLD (3 and 4)

(1) Dipartimento “Scienze di Base e Applicate per l’Ingegneria”,

SAPIENZA - Università di Roma, 16, Via A. Scarpa, 00161 Rome, Italy

E-mail: sandra.carillo@sbai.uniroma1.it

URL: <http://www.sbai.uniroma1.it/~sandra.carillo>

E-mail: mauro.loschiavo@sbai.uniroma1.it

URL: <http://www.sbai.uniroma1.it/~mauro.loschiavo>

(2) I.N.F.N. - Sez. Roma1, Gr. IV - Mathematical Methods in NonLinear Physics, Rome, Italy

(3) Department of Science Education and Mathematics

Mid Sweden University, S-851 70 Sundsvall, Sweden

E-mail: Cornelia.Schiebold@miun.se

URL: <http://www.miun.se/personal/corneliaschiebold>

(4) Instytut Matematyki, Uniwersytet Jana Kochanowskiego w Kielcach, Poland

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Abstract. The recursion operators admitted by different operator Burgers equations, in the framework of the study of nonlinear evolution equations, are here considered. Specifically, evolution equations wherein the unknown is an operator acting on a Banach space are investigated. Here, the *mirror* non-Abelian Burgers equation is considered: it can be written as $r_t = r_{xx} + 2r_x r$. The structural properties of the obtained recursion operator are studied; thus, it is proved to be a strong symmetry for the *mirror* non-Abelian Burgers equation as well as to be the hereditary. These results are proved via direct computations as well as via computer assisted manipulations; ad hoc routines are needed to treat non-Abelian quantities and relations among them. The obtained recursion operator generates the *mirror* non-Abelian Burgers hierarchy. The latter, when the unknown operator r is replaced by a real valued function reduces to the usual (commutative) Burgers hierarchy. Accordingly, also the recursion operator reduces to the usual Burgers one.

1 Introduction

Non-Abelian Burgers equations are here studied. The idea is to construct different non-commutative counterparts of the Burgers equation in a real valued unknown. Indeed, the non-Abelian Burgers (or non-commutative Burgers) as equation usually considered takes the form of the corresponding nonlinear evolution equation, namely $s_t = s_{xx} + 2ss_x$. Here, the *mirror* non-Abelian Burgers equation is considered: it can be written as $r_t = r_{xx} + 2r_x r$. Both these non-Abelian Burgers equation are studied by Kupershmidt in [20] who constructed the whole hierarchies they generate in the case of matrix equations. Notably, here the unknown in the equations under investigation are supposed to be operators on a suitable Banach space. Hence, these unknown cannot be represented via finite dimensional matrices. More precisely, $r(x, t)$ is a bounded linear endomorphism on some Banach space. In applications, choices for the underlying Banach space include sequence spaces and $L_2(\mathbb{R})$, see [9], [10], [29]. On the other hand, the results on non-commutative hierarchies of finite dimensional matrix equations are naturally included, as a particular case, in the present study. Non-Abelian generalization of Burgers equation where the unknown are finite dimensional matrices are constructed in Bruschi, Levi and Ragnisco [21].

The present study is concerned about structural properties of non-Abelian Burgers equations and represents a continuation of the results in [6, 7], where the recursion operator of the non-Abelian Burgers equation is obtained via a Cole-Hopf [11, 19] transformation linking the non-commutative heat equation and the non-commutative Burgers equation. Then, the obtained operator is proved to satisfy all the required algebraic properties to be the hereditary recursion operator which generates the non Abelian Burgers hierarchy. Specifically, it is both a strong symmetry and a hereditary operator. following the same approach, the mirror hierarchy is generated. Notably, it coincides with the mirror hierarchy proposed by Kupershmidt in [20], who constructs a recursive definition of the hierarchies. Here, the hierarchy is recovered on application of the Cole-Hopf transformation viewed as a particular case of Bäcklund transformation and, hence, the results by Fuchssteiner [14] and Fokas and Fuchssteiner [12] referring to Bäcklund transformations and recursion operators can be applied. In particular, the recursion operator of the mirror Non Abelian Burgers equations is obtained, combining the non-commutative Cole-Hopf transformation with the trivial recursion operator admitted by the non-commutative linear heat equation.

The hierarchy of non-commutative Burgers equations (therein termed right-handed) as well as the corresponding recursion operator in [7] were, independently, obtained by Gürses, Karasu and Turhan [18] on application of a method, in [17], based on the Lax pair formulation.

It should be mentioned that the present investigation is part of a wide reasearch program which takes its origins in the study of structural properties of nonlinear evolution equations, where the unknown is a real valued function, and their connection with Bäcklund transformations [24, 3, 13]. In particular, this work continues the study, currently under further development, on non-Abelian nonlinear evolution equations in [2] - [8], [13], [25]- [29].

The material is organized as follows. The opening Section 2 concerns the *mirror non-Abelian Burgers* equation, termed also mirror non-commutative Burgers equation. This equation is linked, via a *mirror* Cole-Hopf transformation to the noncommutative heat equation. The corresponding hierarchy is generated via subsequent applications of the admitted recursion operator, denoted as $\Phi(r)$ which is later shown to be hereditary. Then, all the equations belonging to the *mirror* Burgers hierarchy follow on subsequent applications of the operator $\Phi(r)$. Notably, this *mirror* Burgers hierarchy is the same obtained by Kupershmidt [20].

In the subsequent Section 3, the obtained operator $\Phi(r)$, is proved to represent a strong symmetry admitted by the mirror non-Abelian Burgers equation.

Sections 4 is devoted the hereditarines of the recursion operator $\Phi(r)$. Notably, there are different ways to prove the hereditariness of the recursion operator $\Phi(r)$. Indeed, as already pointed out, its construction via the Cole-Hopf transformation which links the mirror non-Abelian Burgers equation to the non-commutative linear heat equation indicates it inherits such a property. Furthermore, the result can be proved via a direct computation, following the lines of the proof given in [7] where the hereditariness of the recursion operator admitted by the usual non-Abelian Burgers equation is shown. In addition, the proof can be constructed via a computer assisted method: this is presented in Section 5 where the difficulties which arise when a computer algebra language is used in dealing with non-commutative quantities is pointed out.

Finally, an Appendix devoted to a brief summary on the connection between recursion operators and Bäcklund transformations and, in particular, to summarize those results needed throughout the other Sections, closes this work. In addition, a brief overview on results previously obtained on the noncommutative Burgers equation, its recursion operator and the link with the heat equation are also recalled to help the reader.

2 The *mirror non-Abelian Burgers* hierarchy

In this Section, the non-Abelian Cole-Hopf transformation $r = u_x u^{-1}$ is applied to the heat equation to obtain a *mirror* non-Abelian Burgers equation, as, according to [20], we term it. That is, consider the Bäcklund transformation:

$$B(u, r) = 0, \quad \text{where} \quad B(u, r) = ru - u_x \quad (1)$$

which links the heat equation $u_t = u_{xx}$ to the *mirror* non-Abelian Burgers equation

$$r_t = r_{xx} + 2r_x r, \quad (2)$$

where, following the method in [7], $r_{xx} + 2r_x r = \Phi(r)r_x$, when $\Phi(r)$ denotes the recursion operator admitted by the mirror non-Abelian Burgers equation.

Proposition 1. *The operator $\Phi(r)$ is given by*

$$\Phi(r) = (D - C_r)(D + R_r)(D - C_r)^{-1}, \quad \text{where} \quad C_r := [r, \cdot], \quad (3)$$

i.e. C_r denotes the commutator with r , and R_r is the right multiplication by r .

Proof Given the Cole-Hopf transformation B in (1), its directional derivatives are:

$$\begin{aligned} B_u[V] &= \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \left(r(u + \epsilon V) - (u + \epsilon V)_x \right) = rV - V_x, \\ B_r[W] &= \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \left((r + \epsilon W)u - u_x \right) = Wu, \end{aligned}$$

namely, for $V \in T_u \mathcal{U}$, $W \in T_r \mathcal{S}$, it follows $B_u = L_r - D$ and $B_r = R_u$, hence the transformation operator $T = -B_r^{-1} B_u$. Then, when L_r denotes the left multiplication by r , the following identities

$$\begin{aligned} DR_u &= R_u D + R_u R_r = R_u (D + R_r) \\ L_u D L_{u^{-1}} &= L_u (L_{u^{-1}} D - L_{u^{-1}} L_{u_x} L_{u^{-1}}) = (D - L_r) \\ (D - L_r) R_u &= (R_u D + R_u R_r) - L_r R_u = R_u (D - C_r) \\ L_u (D + R_r) &= D L_u - L_r L_u + L_u R_r = (D - C_r) L_u \\ R_{u^{-1}} D R_u &= D + R_{u^{-1}} R_{u_x} = (D + R_r) \end{aligned}$$

allow to write the transformation operator T in the form

$$T = (D - C_r) R_{u^{-1}}; \quad (4)$$

Then, the recursion operator $\Phi(r)$, given in (3), is obtained via

$$\Phi = T D T^{-1},$$

where D is the *trivial* recursion operator admitted by the linear heat equation. \square

Hence, the mirror non-Abelian Burgers hierarchy is represented by

$$r_{t_n} = \Phi(r)^{n-1} r_x, \quad n \geq 1, \quad (5)$$

the lowest members of which read

$$\begin{aligned} r_{t_1} &= r_x, \\ r_{t_2} &= r_{xx} + 2r_x r, \\ r_{t_3} &= r_{xxx} + 3r_{xx} r + 3r_x^2 + 3r_x r^2. \end{aligned} \quad (6)$$

Note that all the members of this hierarchy are obtained from the corresponding ones in the non-Abelian Burgers hierarchy when left multiplication is replaced with right multiplication. Furthermore, also in this case, the whole hierarchy is linked via a Cole-Hopf mirror transformation, which now is (1), instead of $B(u, s) = us - u_x$. Transformation (1) connects corresponding members in the heat hierarchy (23) to corresponding ones in the non-Abelian Burgers mirror hierarchy (5).

The next Sections are devoted to state and prove the main Theorem on properties of the operator $\Phi(r)$.

3 The non-Abelian mirror Burgers recursion operator

This Section is devoted to the operator $\Phi(r)$ and, in particular, the following theorem is the main result.

Theorem 2. *The operator given in (3), i.e.*

$$\Phi(r) = (D - C_r)(D + R_r)(D - C_r)^{-1}$$

represents the hereditary recursion operator of the non-Abelian mirror Burgers equation.

To prove the Theorem 2 the following steps are needed

- prove that the operator $\Phi(r)$ is a strong symmetry for the base member hierarchy, i.e. $r_t = H(r)$, where $H(r) = r_x$;
- prove that the operator $\Phi(r)$ is hereditary.

Then, combination of the two steps completes the proof since it allows to conclude, as in [5], that the operator $\Phi(r)$ is hereditary. Then, according to [5], it is a strong symmetry for all the higher order nonlinear evolution equations of the non-Abelian mirror Burgers hierarchy (5).

Proof (of Theorem 2) **Step 1** is represented by the following

Proposition 3. *The operator $\Phi(r)$ is a strong symmetry for $r_t = H(r)$, where $H(r) = r_x$.*

Proof (of Proposition 3) The proposition is proved when ¹ the condition

$$\Phi'(r)[H(r)] = [H', \Phi(r)] \tag{7}$$

is shown to hold.

First of all, note that since $H(r) = r_x$, then $H'(r) = D$; thus, on substitution of both of them the relation to prove becomes

$$\Phi'(r)[r_x] = [D, \Phi(r)]. \tag{8}$$

For computational convenience, the operator $\Phi(r)$ is re-written in the equivalent form

$$\Phi(r) = D + R_r + L_{r_x}(D - C_r)^{-1} \tag{9}$$

where, respectively, C_r , R_r and L_r denote the commutator, right and left multiplication by r , that is

$$C_r\sigma := [r, \sigma], \quad R_r\sigma := \sigma r, \quad L_r\sigma := r\sigma, \quad \forall \sigma.$$

¹see the Appendix and [14].

Direct computation proves the thesis. The Fréchet derivatives of the operator $\Phi(r)$, in (9), is

$$\Phi'(r)[V] = R_V + L_{V_x}(D - C_r)^{-1} + L_{r_x}(D - C_r)^{-1}C_V(D - C_r)^{-1}. \quad (10)$$

The latter follows since $\forall V$, $C'_r[V] = C_V$, $R'_r[V] = R_V$, $L'_{r_x}[V] = L_{V_x}$, and product rule is applied so that the Fréchet derivative of $(D - C_r)^{-1}$ follows²

$$((D - C_r)^{-1})'[V] = (D - C_r)^{-1}C_V(D - C_r)^{-1}. \quad (11)$$

To evaluate $\Phi'(r)[r_x]$, let $V = r_x$ in (10),

$$\Phi'(r)[r_x] = R_{r_x} + L_{r_{xx}}(D - C_r)^{-1} + L_{r_x}(D - C_r)^{-1}C_{r_x}(D - C_r)^{-1}. \quad (12)$$

Now, since $[D - C_r, D] = [D, C_r] = C_{r_x}$ implies $[D, (D - C_r)^{-1}] = (D - C_r)^{-1}C_{r_x}(D - C_r)^{-1}$, the right hand side gives

$$\begin{aligned} [D, \Phi(r)] &= [D, R_r] + [D, L_{r_x}(D - C_r)^{-1}] \\ &= R_{r_x} + L_{r_{xx}}(D - C_r)^{-1} + L_{r_x}[D, (D - C_r)^{-1}] \\ &= R_{r_x} + L_{r_{xx}}(D - C_r)^{-1} + L_{r_x}(D - C_r)^{-1}C_{r_x}(D - C_r)^{-1}, \end{aligned} \quad (13)$$

Comparison of (12) with (13) shows (8) and completes the proof. \square

The next **Step 2** needed to prove Theorem 2 is represented by the proof that the operator $\Phi(r)$ is hereditary: this result is established in the next Section.

Remark A computer algebra program (using a symbolic language) was constructed to provide a computer assisted proof of the recursivity of the operator $\Phi(r)$ and the hereditariness of the same operator. Note that one of the main difficulties to overcome writing computer routines that may prove results concerning non-Abelian properties is that in the symbolic language, by default, all the variables are assumed to commute. Hence, non commutativity requires non trivial *ad hoc* routines.

On the other hand, in devising the computer assisted proof there is no need of introducing the notion of equivalence between operators, a relation useful to simplify the computations done by hand. For instance, in [7], equivalence relations are introduced to avoid the explicit computation of those terms whose contribution satisfies the due symmetry requirement. The computer algebra routines we prepared straightly produces all the terms and, then, verify symmetry after the exchange of the two arbitrary fields therein and the consequent sum. Some of the details are given in Section 5. \square

4 The hereditariness of the non-commutative mirror Burgers recursion operator

This section is devoted to the hereditariness of non-commutative mirror Burgers recursion operator. The definition of hereditariness, introduced in [14] in the context of nonlinear evolution equations, represents a key tool since, a strong symmetry (recursion operator according to [22]) which is also hereditary represents a strong symmetry also for each one the nonlinear evolution equations of the hierarchy, in this case (5), it generates. That is, the property is inherited from one equation to the next one in the hierarchy and, hence, to the whole hierarchy. Hereditariness (see the definition in the Appendix) is an algebraic property: it can be verified when a bilinear form is checked to be symmetric with respect to the exchange between each other of two arbitrary chosen fields it acts on.

²Recall that $(\Gamma^{-1}(r))'[V] = \Gamma(r)^{-1}(-\Gamma'(r)[V])\Gamma(r)^{-1}$ holds for an operator-valued function $\Gamma(r)$.

Theorem 4. (Statement) *The non-commutative mirror Burgers recursion operator given in (3) is hereditary.*

The thesis of this Theorem can be proved in various different ways.

1. Indeed, Fokas and Fuchssteiner [12] proved that hereditary operators are mapped to hereditary operators via Bäcklund transformations. This result can be applied to the non-commutative mirror Burgers recursion operator since it is obtained via the Cole-Hopf transformation of the trivial recursion operator D admitted by the heat equation.
2. Following the method in [7], a direct proof can be constructed computing all the terms in (17). Note that, the notion of equivalence can be introduced to simplify the required computations.
3. In addition, via an *ad hoc* computer algebra program which verifies that (17) holds true.

This third choice is examined in the next Section. Note that the idea to employ computer algebra routines to investigate properties of recursion operators is not new, see [16] for early results, and [1] (and references therein) for recent developments on the subject. However, all of them are concerned about non linear evolution equations where the unknown is a real valued function and hence, the devised routines, in different symbolic languages, are in an Abelian framework while the present investigation concerns non-Abelian operator unknowns.

5 Computer assisted results

To ease-up the proof of some of the analytic properties of recursion operators, a computer algebra program (using a symbolic language) was constructed that provides an automatic assisted achievement of the necessary steps. At first, proof of the recursivity of the operator $\Phi(r)$ has been produced. Then, to also prove hereditariness of the same operator a second computer algebra program has been realized. Clearly, computer algebra is convenient when long and tedious computations are necessary, however it must be noticed that other technical problems arise. For instance, one of the main difficulties to overcome has been that of writing routines that proved results concerning non-Abelian computations. Indeed in the symbolic language, by default, all the variables are assumed to commute, and all the operations such as multiplications, derivatives, and similar, are commutative by default. Hence, non commutativity required non trivial *ad hoc* routines.

Specifically, automatic proofs procedure developed along the following subsequent steps.

The first step concerned realizing that operator $\Phi(r)$ may be easily rewritten if a convenient *derivation* is introduced, namely, let us introduce the operator: $\mathbb{D} := (D - C_r)$. Indeed, \mathbb{D} has all necessary and characteristic properties of a derivative (linearity, Leibnitz rule, etc.), and in the course of computation it may be (and has been) used and interpreted as a normal derivative, provided that its real meaning is kept into account. This is not only to say that: when $\mathbb{D}f(x(\eta))$ needs to be computed then the result is $f_x - [r, f]$, but also that, when any other algebraic rule is concerned, the new derivative $\mathbb{D} =: \frac{\partial}{\partial \eta}$ may replace the former $\frac{\partial}{\partial x} = D$ derivative, until the variable $x \in \mathbb{R}$ is replaced back at its place.

The second step is then that of writing the operator $\Phi(r)$ by use of this new convenient derivative \mathbb{D} . Its consequent compact form, from (3), is easily found to be:

$$\Phi(r) = \mathbb{D}(\mathbb{D} + L_r)\mathbb{D}^{-1}$$

and since it clearly is $\mathbb{D}r \equiv Dr$, then this compact form for $\Phi(r)$ immediately shows that the equation's hierarchy is simply given by

$$\Phi^n(r)Dr = \mathbb{D}(\mathbb{D} + L_r)^n r .$$

In particular, the compact form for the *mirror* Burgers equation has the easy aspect

$$r_t = \Phi(r)Dr = \mathbb{D}(\mathbb{D} + L_r)r = \mathbb{D}^2r + \mathbb{D}r^2 .$$

Third step has been that of confirming the recursivity property of $\Phi(r)$ by automatic computation with use of this new operator \mathbb{D} . To achieve this, its Fréchet derivative is needed, yet obviously keeping in mind that \mathbb{D} is still a function of the equation variable r , and hence that the following hold

$$\begin{cases} (\mathbb{D})'[V] = (D - C_r)'[V] = -C_V \\ (\mathbb{D}^{-1})'[V] = -\mathbb{D}^{-1}(\mathbb{D})'\mathbb{D}^{-1} = (D - C_r)^{-1}C_V(D - C_r)^{-1} . \end{cases}$$

Consequently, the Fréchet derivative of $\Phi(r) = D + R_r + L_{r_x}(D - C_r)^{-1}$, given in (10), that is

$$\Phi'[V] = R_V + L_{V_x}(D - C_r)^{-1} + L_{r_x}((D - C_r)^{-1})'[V] ,$$

turns out to acquire the computational more convenient form

$$\Phi'[V] = -C_V + L_V + L_{V_x}\mathbb{D}^{-1} + L_{r_x}\mathbb{D}^{-1}C_V\mathbb{D}^{-1} \quad (14)$$

$$= R_V + L_{\mathbb{D}V}\mathbb{D}^{-1} + L_{[r,V]}\mathbb{D}^{-1} + L_{r_x}\mathbb{D}^{-1}C_V\mathbb{D}^{-1} , \quad (15)$$

where it must be recalled that the field $\mathbb{D}V$ is in fact $\frac{\partial}{\partial \eta} V = \frac{\partial}{\partial x} V - C_r V$.

Next step is that of the technical (long and tedious) computations of the desired properties. The first one, recursivity, is first performed using the base member of the hierarchy, according to which the condition $\Phi'[r_x] - [D, \Phi] = 0$ is verified. To prove this fact, the operator \mathbb{D} may be used as the (unique) derivative operator with respect to the *new* variable η , however it has still been kept in mind that this is possible only by replacing the *old* derivative $\frac{\partial}{\partial x} = D$ by the operator $(\mathbb{D} + C_r)$, and by using the Fréchet derivative of $\Phi(r)$ with its form (15). This is actually what it has been done to confirm the explicit direct proof that is also provided in the previous Section. Furthermore, also to check the automatic procedure, the next hierarchy member has been obtained:

$$\Phi'[H(r)] = [H'(r), \mathbb{D}(\mathbb{D} + L_r)\mathbb{D}^{-1}] \quad (16)$$

where $H(r)$ is the (symmetric) Burgers equation: $H(r) = \mathbb{D}^2r + \mathbb{D}r^2$, and H' is its r -derivative: $H'(r) = D^2 + 2R_rD + 2L_{r_x}$ expressed in the new coordinates (and remember that $r_\eta \equiv r_x$):

$$H'(r) = \mathbb{D}^2 - R_{r_\eta} + 3L_{r_\eta} + L_{r^2} - R_{r^2} + 2L_r\mathbb{D} .$$

It is useful to remark here that, although the variable η coincides with the variable x , all the same, due to non-commutative asset, their two derivations \mathbb{D} and D are different, and may coincide only in the commutative case. Only for convenience, we write here the common value of (16):

$$\begin{pmatrix} r_{\eta\eta} \\ r r_\eta \\ R_{r_{\eta\eta}}\mathbb{D}^{-1} \\ r_\eta r \\ 2R_r R_{r_{\eta\eta}}\mathbb{D}^{-1} \\ 2R_{r_\eta} R_{r_\eta}\mathbb{D}^{-1} \\ R_r R_r R_{r_\eta}\mathbb{D}^{-1} \end{pmatrix} + \begin{pmatrix} R_{r_\eta}\mathbb{D}^{-1}R_{r_{\eta\eta}}\mathbb{D}^{-1} \\ -R_{r_\eta}R_r R_{r_\eta}\mathbb{D}^{-1} \\ -R_{r_\eta}\mathbb{D}^{-1}r_{\eta\eta}\mathbb{D}^{-1} \\ R_{r_\eta}\mathbb{D}^{-1}R_r R_{r_\eta}\mathbb{D}^{-1} \\ R_{r_\eta}\mathbb{D}^{-1}R_{r_\eta} R_r\mathbb{D}^{-1} \\ -R_{r_\eta}\mathbb{D}^{-1}r r_\eta\mathbb{D}^{-1} \\ -R_{r_\eta}\mathbb{D}^{-1}r_\eta r\mathbb{D}^{-1} \end{pmatrix}$$

It may also be remarked that, although this being only a matter of chance, the same result may be found if the derivative \mathbb{D} is not considered as a function of r itself, but only as a *single* derivative. It is in fact immediate to see that in this case:

$$\Phi'[r_\eta] - [\mathbb{D}, \Phi] = \mathbb{D}L_{r_\eta}\mathbb{D}^{-1} + \mathbb{D}(\mathbb{D} + L_r) - \mathbb{D}^2(\mathbb{D} + L_r)\mathbb{D}^{-1} = 0.$$

Unluckily, this fortunate event does not repeat itself in the more difficult case of hereditariness. Indeed, to prove that operator $\Phi(r)$ is hereditary the complete form (15) must be used, and a long computation is necessary, together with several integration by parts, to acquire the desired result. In fact, if the difference $\Phi\Phi'[V] - \Phi'[\Phi V]$ is subdivided into its four terms due to the four terms of operator Φ' , namely:

$$\begin{aligned} \Phi'_1[V] &= R_V & \Phi'_3[V] &= C_r V \mathbb{D}^{-1} \\ \Phi'_2[V] &= L_{\mathbb{D}V} \mathbb{D}^{-1} & \Phi'_4[V] &= r_\eta \mathbb{D}^{-1} C_V \mathbb{D}^{-1} \end{aligned}$$

then the four values for the difference $S_j := \Phi\Phi'_j[V] - \Phi'_j[\Phi V]$, $j = 1, \dots, 4$ are as follows ³

$$\begin{aligned} S_1 &= \begin{pmatrix} -R_r V \\ -R_{r_\eta}(\mathbb{D}^{-1}V) \\ R_V \mathbb{D} \\ r R_V \\ r_\eta \mathbb{D}^{-1} R_V \end{pmatrix} & S_2 &= \begin{pmatrix} V_\eta \\ -2r_\eta V \mathbb{D}^{-1} \\ -r_{\eta\eta}(\mathbb{D}^{-1}V) \mathbb{D}^{-1} \\ r_\eta \mathbb{D}^{-1} V_\eta \mathbb{D}^{-1} \end{pmatrix} \\ S_3 &= \begin{pmatrix} r V \\ -V r \\ r_\eta V \mathbb{D}^{-1} \\ -V r_\eta \mathbb{D}^{-1} \\ r_\eta(\mathbb{D}^{-1}V) r \mathbb{D}^{-1} \\ -r r_\eta(\mathbb{D}^{-1}V) \mathbb{D}^{-1} \\ r_\eta \mathbb{D}^{-1} r V \mathbb{D}^{-1} \\ -r_\eta \mathbb{D}^{-1} V r \mathbb{D}^{-1} \end{pmatrix} & S_4 &= \begin{pmatrix} r_\eta V \mathbb{D}^{-1} \\ -r_\eta R_V \mathbb{D}^{-1} \\ r_\eta \mathbb{D}^{-1} R_{V_\eta} \mathbb{D}^{-1} \\ r_\eta \mathbb{D}^{-1} R_{rV} \mathbb{D}^{-1} \\ r_{\eta\eta} \mathbb{D}^{-1} V \mathbb{D}^{-1} \\ -r_\eta \mathbb{D}^{-1} V_\eta \mathbb{D}^{-1} \\ -r_{\eta\eta} \mathbb{D}^{-1} R_V \mathbb{D}^{-1} \\ r r_\eta \mathbb{D}^{-1} V \mathbb{D}^{-1} \\ -r r_\eta \mathbb{D}^{-1} R_V \mathbb{D}^{-1} \\ -r_\eta \mathbb{D}^{-1} r V \mathbb{D}^{-1} \\ -r_\eta \mathbb{D}^{-1} r_\eta(\mathbb{D}^{-1}V) \mathbb{D}^{-1} \\ r_\eta \mathbb{D}^{-1} r_\eta \mathbb{D}^{-1} V \mathbb{D}^{-1} \\ -r_\eta \mathbb{D}^{-1} r_\eta \mathbb{D}^{-1} R_V \mathbb{D}^{-1} \\ r_\eta \mathbb{D}^{-1} R_{r_\eta}(\mathbb{D}^{-1}V) \mathbb{D}^{-1} \end{pmatrix} \end{aligned}$$

It is clear that even in the automatic procedure the form (15) implies that the actual result of the term that has to be symmetric in the exchange $W \leftrightarrow V$ is sufficiently long:

³In the following terms, the symbol L , which denotes left multiplication, is omitted to simplify the notation.

$$\begin{pmatrix}
 r_\eta \mathbb{D}^{-1}(\mathbb{D}^{-1}W) V_\eta \\
 r_\eta \mathbb{D}^{-1}W V \\
 r_\eta \mathbb{D}^{-1}r_\eta (\mathbb{D}^{-1}V (\mathbb{D}^{-1}W)) \\
 r_\eta \mathbb{D}^{-1}(\mathbb{D}^{-1}W) r V \\
 r_\eta \mathbb{D}^{-1}(\mathbb{D}^{-1}W) r_\eta (\mathbb{D}^{-1}V) \\
 W_\eta V \\
 V_\eta W \\
 r_{\eta\eta} \mathbb{D}^{-1}V (\mathbb{D}^{-1}W) \\
 -r_\eta \mathbb{D}^{-1}r_\eta (\mathbb{D}^{-1}(\mathbb{D}^{-1}W) V) \\
 -r_\eta \mathbb{D}^{-1}V r (\mathbb{D}^{-1}W) \\
 -r_\eta \mathbb{D}^{-1}r_\eta (\mathbb{D}^{-1}V) (\mathbb{D}^{-1}W) \\
 -r_{\eta\eta} \mathbb{D}^{-1}(\mathbb{D}^{-1}W) V
 \end{pmatrix} + \begin{pmatrix}
 r W V \\
 r V W \\
 r r_\eta \mathbb{D}^{-1}V (\mathbb{D}^{-1}W) \\
 -r r_\eta \mathbb{D}^{-1}(\mathbb{D}^{-1}W) V \\
 -W r V \\
 -W r_\eta (\mathbb{D}^{-1}V) \\
 -V r_\eta (\mathbb{D}^{-1}W) \\
 -V r W \\
 -r_\eta (\mathbb{D}^{-1}W) V \\
 -r_{\eta\eta} (\mathbb{D}^{-1}V) (\mathbb{D}^{-1}W) \\
 r_\eta (\mathbb{D}^{-1}V) r (\mathbb{D}^{-1}W) \\
 -r r_\eta (\mathbb{D}^{-1}V) (\mathbb{D}^{-1}W)
 \end{pmatrix}$$

The final step in the automatic computations has been that of proving that this term is indeed symmetric in the exchange between W and V , a fact that as already mentioned has required several integrations by part, many of which proved to be more conveniently solved by hand rather than by the automatic procedure.

On the other hand, when the necessary *macros* for the symbolic language are ready for the non-commutative Burgers' mirror equation, then it is only a matter of care to use them again with some similar equation. For instance, all the corresponding properties of the direct non-commutative Burgers' equation: $s_t = \Psi(s)s_x = s_{xx} + 2ss_x$ have again been found with respect to the corresponding *new* derivation: $\mathbb{D} := (D + C_s)$.

6 Appendix

The aim of this Appendix is twofold; indeed, it collects, in its initial part, some background notions and definitions used throughout the whole article while, in the second part, results on the non-Abelian Burgers, in [7] are briefly recalled.

6.1 some background definitions

Definition 1. (Symmetry)

Given an evolution equation $u_t = K(u)$, where $u(x, \cdot) \in M$ and $K : M \rightarrow TM$ is an appropriate C^∞ vector field on a manifold M , a map $\sigma : M \rightarrow TM$ is said to be an *infinitesimal symmetry generator* (for short symmetry) if it leaves the evolution equation itself invariant under the infinitesimal transformation $u \rightarrow u + \varepsilon\sigma$.

As stated in [14], if σ and K are in involution, i.e. if $[\sigma, K]$ is identically zero, then σ is a symmetry of the given nonlinear evolution equation.

Definition 2. (Strong Symmetry)

An operator-valued function $\Gamma(u)$ is called a strong symmetry of $u_t = K(u)$ if, for every symmetry V it admits, the vector field $\Gamma(u)V$ is again a symmetry.

If Γ is a strong symmetry of $u_t = K(u)$, as proved in [14], the condition $\Gamma'[K]V = K'[\Gamma V] - \Gamma K'[V]$ holds for any vector field V .

Definition 3. (Hereditariness)

An operator-valued function Γ is called hereditary if for every $u \in M$ where Γ is defined, the bilinear form

$$\Gamma \Gamma'[V]W - \Gamma'[V]W, \tag{17}$$

is symmetric in $V, W \in T_u M$.

6.2 The non-Abelian Burgers hierarchy

This Section is devoted to a brief overview on known results concerning the non-Abelian Burgers equation, the related recursion operator as well as the hierarchy it generates. Crucial tool is a non-Abelian generalization of the Cole-Hopf transformation connecting the Burgers equation to the linear heat equation. Given the non-Abelian heat equation

$$u_t = K(u) \quad , \quad K(u) = u_{xx} \quad (18)$$

and the non-Abelian Burgers equation

$$s_t = G(s) \quad , \quad G(s) = s_{xx} + 2ss_x, \quad (19)$$

they are connected via the Cole-Hopf transformation $s = u^{-1}u_x$, which can be written under the form of Bäcklund transformation:

$$B(u, s) = 0 \quad , \quad \text{where} \quad B(u, s) = us - u_x \quad . \quad (20)$$

This connection, given the trivial recursion operator D , admitted by the heat equation, according to [6, 7], allows to construct the recursion operator $\Psi(s)$, admitted by the Burgers equation, that is

$$\Psi(s) = (D + C_s)(D + L_s)(D + C_s)^{-1}, \quad (21)$$

which can also be written as

$$\Psi(s) = D + L_s + R_{s_x}(D + C_s)^{-1}. \quad (22)$$

The latter is the form of the recursion operator also obtained by Gürses, Karasu and Turhan [18] via a Lax pair representation of the non-commutative Burgers hierarchy. Then, the following hierarchies, respectively (23) and (24), are constructed on application of the trivial recursion operator D , admitted by the heat equation and the recursion operator $\Psi(s)$ in (21)

$$u_{t_n} = D^{n-1}u_x, \quad n \geq 1, \quad (23)$$

the lowest members of which read

$$\begin{aligned} u_{t_1} &= u_x, \\ u_{t_2} &= u_{xx}, \\ u_{t_3} &= u_{xxx}. \end{aligned}$$

and

$$s_{t_n} = \Psi(s)^{n-1}s_x, \quad n \geq 1, \quad (24)$$

the lowest members of which read

$$\begin{aligned} s_{t_1} &= s_x, \\ s_{t_2} &= s_{xx} + 2ss_x, \\ s_{t_3} &= s_{xxx} + 3ss_{xx} + 3s_x^2 + 3s^2s_x. \end{aligned}$$

The algebraic properties of the operator $\Psi(s)$, firstly obtained in [6] and, independently, in [18], are studied in [7] where $\Psi(s)$ is proved to be a strong symmetry, which is also hereditary.

Remark Finally, note that, as expected, if the unknown operator functions s and r , respectively, in the non-Abelian Burgers (24) and mirror non-Abelian Burgers hierarchy (5) are replaced by a real valued unknown function v , then, the commutative Burgers hierarchy is obtained. Furthermore, when v is substituted to s and r , in turn, in the expressions of the two recursion operators $\Psi(s)$, in (22), and $\Phi(r)$, in (9), they both reduce to the usual (commutative) form of the Burgers hereditary recursion operator, that is

$$\Phi(v) \equiv \Psi(v) = D + v + v_x D^{-1}. \quad (25)$$

Hence, the (commutative) Burgers hierarchy follows as a special case of both the non-Abelian Burgers hierarchies (24) and (5).

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